Exercise 25

Solve the differential equation using the method of variation of parameters.

$$y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 3y_c' + 2y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 3(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 3r + 2 = 0$$

Solve for r.

$$(r-2)(r-1) = 0$$

$$r = \{1, 2\}$$

Two solutions to the ODE are e^x and e^{2x}. By the principle of superposition, then,

$$y_c(x) = C_1 e^x + C_2 e^{2x}.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 3y_p' + 2y_p = \frac{1}{1 + e^{-x}} \tag{2}$$

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x)e^x + C_2(x)e^{2x}$$

Differentiate it with respect to x.

$$y_p' = C_1'(x)e^x + C_2'(x)e^{2x} + C_1(x)e^x + 2C_2(x)e^{2x}$$

If we set

$$C_1'(x)e^x + C_2'(x)e^{2x} = 0, \tag{3}$$

then

$$y_p' = C_1(x)e^x + 2C_2(x)e^{2x}.$$

Differentiate it with respect to x once more.

$$y_p'' = C_1'(x)e^x + 2C_2'(x)e^{2x} + C_1(x)e^x + 4C_2(x)e^{2x}$$

Substitute these formulas into equation (2).

$$\left[C_1'(x)e^x + 2C_2'(x)e^{2x} + \cancel{C_1(x)e^x} + \cancel{4C_2(x)e^{2x}}\right] - 3\left[\cancel{C_1(x)e^x} + \cancel{2C_2(x)e^{2x}}\right] + 2\left[\cancel{C_1(x)e^x} + \cancel{C_2(x)e^{2x}}\right] = \frac{1}{1+e^{-x}}$$

Simplify the result.

$$C_1'(x)e^x + 2C_2'(x)e^{2x} = \frac{1}{1+e^{-x}} \tag{4}$$

Subtract the respective sides of equations (3) and (4) to eliminate $C_1'(x)$.

$$C_2'(x)e^{2x} = \frac{1}{1+e^{-x}}$$

Solve for $C_2'(x)$.

$$C_2'(x) = \frac{e^{-2x}}{1+e^{-x}}$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$\begin{aligned}
C_2(x) &= \int^x C_2'(w)\,dw \\
&= \int^x \frac{e^{-2w}}{1+e^{-w}}\,dw \\
&= \int^x \frac{1}{e^w(e^w+1)}\,dw \\
&= \int^{e^x} \frac{1}{u(u+1)}\left(\frac{du}{u}\right) \\
&= \int^{e^x} \frac{1}{u^2(u+1)}\,du \\
&= \int^{e^x} \left(\frac{1}{u^2} - \frac{1}{u} + \frac{1}{u+1}\right) du \\
&= \left(-\frac{1}{u} - \ln|u| + \ln|u+1|\right)\Big|^{e^x} \\
&= \left(-\frac{1}{u} + \ln\left|\frac{u+1}{u}\right|\right)\Big|^{e^x} \\
&= \left(-\frac{1}{u} + \ln\left|1+\frac{1}{u}\right|\right)\Big|^{e^x} \\
&= -e^{-x} + \ln(1+e^{-x})
\end{aligned}$$

Solve equation (3) for $C_1'(x)$.

$$\begin{aligned} C_1'(x) &= -C_2'(x)e^x \\ &= -\left(\frac{e^{-2x}}{1+e^{-x}}\right)e^x \\ &= -\frac{e^{-x}}{1+e^{-x}} \end{aligned}$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$\begin{aligned} C_1(x) &= \int^x C_1'(w)\,dw \\ &= -\int^x \frac{e^{-w}}{1+e^{-w}}\,dw \\ &= \int^{1+e^{-x}} \frac{du}{u} \\ &= \ln|u|\Big|^{1+e^{-x}} \\ &= \ln|1+e^{-x}| \\ &= \ln(1+e^{-x}) \end{aligned}$$

Therefore,

$$\begin{aligned} y_p &= C_1(x)e^x + C_2(x)e^{2x} \\ &= \ln(1+e^{-x})e^x + [-e^{-x} + \ln(1+e^{-x})]e^{2x} \\ &= (e^x + e^{2x})\ln(1+e^{-x}) - e^x, \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^x + C_2e^{2x} + (e^x + e^{2x})\ln(1+e^{-x}) - e^x \\ &= (C_1 - 1)e^x + C_2e^{2x} + (e^x + e^{2x})\ln(1+e^{-x}) \\ &= C_3e^x + C_2e^{2x} + (e^x + e^{2x})\ln(1+e^{-x}), \end{aligned}$$

where C_2 and C_3 are arbitrary constants.